thrust solutions. Finally, we note that except for the above six points (two low-thrust, four impulsive) and one additional singularity at 270 deg there exists a finite solution for $\bar{a}$ at every ψ. However, on some intervals such as [206 deg, 227 deg] the acceleration level is above 6.0 g's, and for all practical purposes these must be considered impulsive solutions.

Another interesting feature is the existence of four minima (discounting the zero minima) and one maximum (the one 270 deg is infinite). What is particularly interesting is that the maximum at 90 deg has a value very nearly equal to, but not exactly equal to, unity. (The value is about 0.976.) The two smaller minima have values around 0.2, while the larger minima have values of about 4.64. Again, this last value is sufficiently large to be considered an impulsive solution. Thus the two minima at $\psi = 37$ and $\psi = 143$ and the single maximum at 90 deg are the three most interesting points. Detailed exploration of the trajectories corresponding to these is left to another paper, but it appears that $\psi = 90$ deg is associated with the circularization of a straight-in parabolic trajectory ($R_p = 0$, $R_a = \infty$), while the two minima are associated with Hohmann-type maneuvers.

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Optimal Low-Thrust Maneuvers Near the Libration Points

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Introduction

As has been previously reported[1] an investigation was made of the minimum propellant optimal maneuvers near the libration points in the Earth-moon system of a space vehicle equipped with a low-thrust propulsion installation. The study was made only for collinear libration points on the basis of some approximate solutions of the differential equations of the extremals ($\omega = 0$). The present paper carries the work further and gives this study a complete form on the basis of some rigorous solutions both for collinear and equidistant libration points ($\omega \neq 0$).

The system of units and the notations in this Note are those used in Ref. 1.

Variational Problem

In Ref. 1 it was shown that generally the variational problem of the minimum propellant optimal maneuvers near the libration points is one of extremum with constraints. On introducing Lagrange's multipliers λ_1, λ_2, λ_3, λ_4, and reducing the problem of extremum with constraints to one of extremum without constraints, after performing the calculations we get the following system of differential equations of the extremals:

$$\frac{d\xi}{dt} = V_\xi \qquad \frac{d\eta}{dt} = V_\eta \tag{1a}$$

$$\frac{dV_\xi}{dt} = 2\omega V_\eta + K_1\xi + K_3\eta + a_\xi \tag{1b}$$

$$\frac{dV_\eta}{dt} = -2\omega V_\xi + K_2\eta + K_3\xi + a_\eta \tag{1c}$$

$$\frac{d\lambda_1}{dt} = -K_1\lambda_2 - K_3\lambda_4 \qquad \frac{d\lambda_2}{dt} = -\lambda_1 + 2\omega\lambda_4 \tag{1d}$$

$$\frac{d\lambda_3}{dt} = -K_3\lambda_2 - K_2\lambda_4 \qquad \frac{d\lambda_4}{dt} = -2\omega\lambda_2 - \lambda_3 \tag{1e}$$

where the quantities K_1, K_2, K_3 are given in Ref. 1; and the algebraic equations:

$$2a_\xi - \lambda_2 = 0 \qquad 2a_\eta - \lambda_4 = 0 \tag{2}$$

Exact Solutions for the Collinear and Equidistant Libration Points

In order to integrate Eqs. (1a-1c), it is necessary first to know the functions a_ξ and a_η which, taking account of Eq. (2), can be obtained by integrating Eqs. (1d) and (1e).

The differential equation system [Eqs. (1d) and (1e)] has the characteristic equation

$$\Lambda^4 + b\Lambda^2 + c = 0 \tag{3}$$

where

$$b = 4\omega^2 - K_1 - K_2 \qquad c = K_1K_2 - K_3^2$$

which can have either two real and two imaginary roots or all four roots imaginary.

We consider first the case where two roots are real and two are imaginary, the case of all roots imaginary being obtained from the first by particularization.

Let

$$\Lambda_{1,2} = \pm p \qquad \Lambda_{3,4} = \pm iq \tag{4}$$

where

$$p^2, q^2 = \tfrac{1}{2}\left(\mp b + \sqrt{b^2 - 4c}\right)$$

Using the current method of integration, the solution of Eqs. (1d) and (1e) can be set after some calculation under the form

$$\lambda_1 = D_1\cosh pt + D_2\sinh pt + D_3\cos qt + D_4\sin qt \tag{5a}$$

$$\lambda_2 = (A_1D_1 + A_2D_2)\cosh pt + (A_2D_1 + A_1D_2)\sinh pt + (A_3D_3 - A_4D_4)\cos qt + (A_4D_3 + A_3D_4)\sin qt \tag{5b}$$

with λ_3 and λ_4 obtained by replacing A_k with B_k and A_k with C_k, respectively ($k = 1,2,3,4$). The quantities A_k, B_k, and C_k are given in Appendix A.

From the above expressions we obtain

$$a_\xi = \tfrac{1}{2}\lambda_2 \qquad a_\eta = \tfrac{1}{2}\lambda_4 \tag{6}$$

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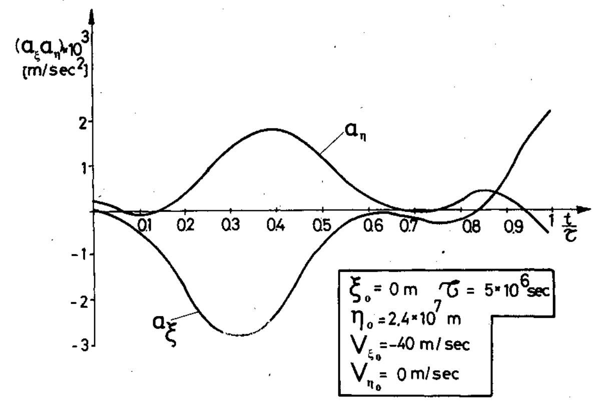

Fig. 1 Components of acceleration due to thrust in $\angle\xi\eta$ system near point L_5 ($\omega\neq 0$).

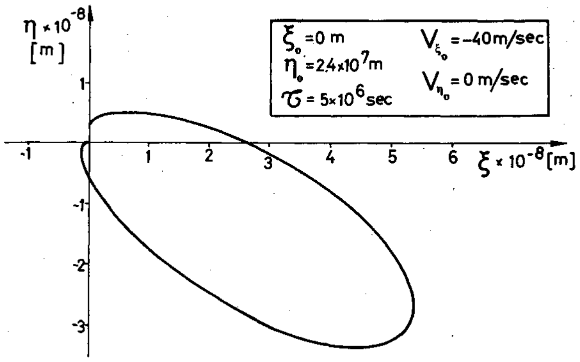

Fig. 2 Optimal trajectory of vehicle near L_5 ($\omega\neq 0$).

with the aid of which the system of differential equations (1a-1c) can be integrated.

Using the method of the variation of constants, the solution of this system can be set, after performing some calculations, under the form

$$\xi = (D_5 + P^1_\xi D_1 + P^2_\xi D_2)\cosh pt + (D_6 + P^2_\xi D_1 + P^1_\xi D_2)\sinh pt$$
$$+ (D_7 + P^3_\xi D_3 + P^4_\xi D_4)\cos qt + (D_8 - P^4_\xi D_3 + P^3_\xi D_4)\sin qt \tag{7a}$$

$$V_\xi = (P^1_{V_\xi} D_1 + P^2_{V_\xi} D_2 + \tilde P^1_{V_\xi} D_5 + \tilde P^2_{V_\xi} D_6)\cosh pt$$
$$+ (P^2_{V_\xi} D_1 + P^1_{V_\xi} D_2 + \tilde P^2_{V_\xi} D_5 + \tilde P^1_{V_\xi} D_6)\sinh pt$$
$$+ (P^3_{V_\xi} D_3 + P^4_{V_\xi} D_4 + \tilde P^3_{V_\xi} D_7 + \tilde P^4_{V_\xi} D_8)\cos qt$$
$$+ (-P^4_{V_\xi} D_3 + P^3_{V_\xi} D_4 - \tilde P^4_{V_\xi} D_7 + \tilde P^3_{V_\xi} D_8)\sin qt \tag{7b}$$

where η is obtained by replacing in Eq. (7b) $P^k_{V_\xi}$ with P^k_η and $\tilde P^k_{V_\xi}$ with $\tilde P^k_\eta$ and V_η by replacing $P^k_{V_\xi}$ with $P^k_{V_\eta}$ and $\tilde P^k_{V_\xi}$ with $\tilde P^k_{V_\eta}$. The quantities P^k_ξ, $\tilde P^k_\xi$, ..., $\tilde P^k_{V_\eta}$ are given in AppendixB.

In order to determine the functions $\xi(t)$, $\eta(t)$, $V_\xi(t)$, $V_\eta(t)$, $a_\xi(t)$, and $a_\eta(t)$ which characterize the minimum propellant optimal maneuvers near the collinear libration points (since for all three collinear points L_1, L_2, and L_3 the conditions $K_1>0$, $K_2<0$, and $K_3=0$ are fulfilled), the characteristic equation (3) similar to that of the homogeneous system obtained from Eqs. (1a-1c) has two real and two imaginary roots. In this case where the functions remain in the form given by Eqs. (5-7), with the terms A_k, B_k, C_k and P^1_ξ, P^2_ξ,...,$\tilde P^4_{V_\eta}$ as given in Appendices A and B, we set $K_3=0$.

For the equidistant libration points L_4 and L_5, the conditions $K_1>0$, $K_2>0$, and $K_3\neq 0$ are fulfilled. In this case all four roots of the characteristic equation are imaginary and the functions $\xi(t)$, $\eta(t)$, ..., $a_\eta(t)$ will be given by Eqs. (5-7) in which in a formal manner sine h and cosine h pass into sine and cosine, respectively. The new expressions of A_k, B_k, C_k, P^1_ξ,P^3_ξ,P^4_ξ,P^1_η,..., $\tilde P^4_V$ are obtained by replacing p with ip and the expressions of P^2_ξ,P^2_η, $P^2_{V_\xi}$, $P^2_{V_\eta}$,$\tilde P^2_\eta$, $\tilde P^2_{V_\xi}$, and $\tilde P^2_{V_\eta}$ are replaced with $-iP^2_\xi(ip)$, $-iP^2_\eta(ip)$, $-iP^2_{V_\xi}(ip)$, etc.

The eight integration constants D_1,D_2,...,D_8 can be determined from the conditions given in Ref. 1.

Numerical Applications

First, a numerical application has been carried out for point L_2 (the Earth-moon system). By means of the initial data from Ref. 1 for a combustion duration $\tau=5\times 10^5$ s we have calculated the exact solutions of the components of acceleration due to thrust a_ξ, a_η function of t/τ. Finally, a numerical application for point L_5 has been made.

By means of the initial data $\xi_0=0$, $V_{\xi 0}=-40$ m/s, $\eta_0=24\times 10^6$ m and $V_{\eta 0}=0$ m/s for a combustion duration of $\tau=10^6$ s, we have calculated a_ξ, a_η, and $\eta=\eta(\xi)$ as shown in Figs. 1 and 2.

Conclusions

On the basis of exact solutions, the investigation described here shows that for point L_2 the influence of the components of the Coriolis acceleration neglected in Ref. 1 is substantial.

For the equidistant libration points, the numerical applications made on point L_5 show for initial data comparable to those on L_2 that the acceleration due to thrust, the duration of maneuvers, and the distance traveled are greater. For duration of the maneuvers $\tau\geq 5\times 10^5$ s the acceleration due to thrust falls in the domain of 10^{-4}-$10^{-6}g$.

Appendix A

In Eqs. (5), we have denoted‡

$$r(x) = x^3 + (4\omega^2 - K_2)x - 2\omega K_3 \qquad (x=p,-p,iq,-iq)$$

$$A_x = (K_2 - x^2)/r(x) \qquad B_x = (K_3 x - 2\omega K_2)/r(x) \qquad C_x = (2\omega x - K_3)/r(x)$$

$$A_1 = (A_p + A_{-p})/2 \qquad A_2 = (A_p - A_{-p})/2 \qquad A_3 = (A_q + A_{-q})/2 \qquad A_4 = -i(A_q - A_{-q})/2$$

and similarly for B_1, B_2, ...,C_4.

Appendix B

In Eqs. (7) we have denoted

$$P^1_\xi = \gamma_p\{2[A_2 + (\tilde B_{-p}C_p - \tilde B_p C_{-p})/2]t + [A_1 + (\tilde B_p C_p + \tilde B_{-p}C_{-p})]/p\} + \delta_q(qA_1 - p\tilde B_4 C_2 + q\tilde B_3 C_1)$$

$$P^2_\xi = \gamma_p\{2[A_1 + (\tilde B_{-p}C_p - \tilde B_p C_{-p})/2]t - [A_2 + (\tilde B_p C_p - \tilde B_{-p}C_{-p})]/p\} + \delta_q(qA_2 - p\tilde B_4 C_1 + q\tilde B_3 C_2)$$

‡For $x=\pm iq$ we have denoted $A_{\pm q}$, $B_{\pm q}$, etc.

$$P_\xi^3=\delta_p\{-pA_3-p\tilde{B}_1C_3-q\tilde{B}_2C_4\}+\gamma_q\{2[A_4-i(\tilde{B}_{-q}C_q-\tilde{B}_qC_{-q})/2]t+[A_3+(\tilde{B}_qC_{-q}+\tilde{B}_{-q}C_q)/2]/q\}$$

$$P_\xi^4=\delta_p\{-pA_4-p\tilde{B}_1C_4+q\tilde{B}_2C_3\}+\gamma_q\{-2[A_3+(\tilde{B}_{-q}C_q+\tilde{B}_qC_{-q})/2]t+[A_4-i(\tilde{B}_qC_q-\tilde{B}_{-q}C_{-q})/2]/q\}$$

where

$$\gamma_p=\frac{1}{8p}\frac{p^2-K_2}{p^2+q^2}\qquad \gamma_q=\frac{1}{8q}\frac{q^2+K_2}{p^2+q^2}\qquad \delta_p=\frac{1}{2p}\frac{p^2-K_2}{(p^2+q^2)^2}\qquad \delta_q=\frac{1}{2q}\frac{q^2+K_2}{(p^2+q^2)^2}$$

Using the expressions

$$E_1(X)=\gamma_p\{(\tilde{X}_p\beta_p-\tilde{X}_{-p}\beta_{-p})t-(\tilde{X}_{-p}\alpha_p+\tilde{X}_p\alpha_{-p})/2p\}$$
$$+\delta_q\{p\tilde{X}_4A_2+q\tilde{X}_3A_1-ip(\tilde{X}_q\tilde{B}_{-q}-\tilde{X}_{-q}B_q)C_2/2+q(\tilde{X}_q\tilde{B}_{-q}+\tilde{X}_{-q}\tilde{B}_q)C_1/2\}$$

$$E_2(X)=\gamma_p\{(\tilde{X}_p\beta_p-\tilde{X}_{-p}\beta_{-p})t-(\tilde{X}_{-p}\alpha_p+\tilde{X}_p\alpha_{-p})/2p\}$$
$$+\delta_q\{p\tilde{X}_4A_1+q\tilde{X}_3A_2-ip(\tilde{X}_q\tilde{B}_{-q}-\tilde{X}_{-q}B_q)C_1/2+q(\tilde{X}_q\tilde{B}_{-q}+\tilde{X}_{-q}B_q)C_2/2\}$$

$$E_3(X)=\delta_p\{-p\tilde{X}_1A_3+q\tilde{X}_2A_4-p(\tilde{X}_p\tilde{B}_{-p}+\tilde{X}_{-p}\tilde{B}_p)C_3/2+q(\tilde{A}_p\tilde{B}_{-p}-\tilde{A}_{-p}\tilde{B}_p)C_4/2\}$$
$$+\gamma_q\{-i(\tilde{X}_q\beta_q-\tilde{X}_{-q}\beta_{-q})t+(\tilde{X}_{-q}\alpha_q+\tilde{X}_q\alpha_{-q})/2q\}$$

$$E_4(X)=\delta_p\{-p\tilde{X}_1A_4-q\tilde{X}_2A_3-p(\tilde{X}_p\tilde{B}_{-p}+\tilde{X}_{-p}B_p)C_4/2-q(\tilde{X}_p\tilde{B}_{-p}-\tilde{X}_{-p}\tilde{B}_{-p})C_3/2\}$$
$$+\gamma_q\{-(\tilde{X}_q\beta_q+\tilde{X}_{-q}\beta_{-q})t-i(\tilde{X}_{-q}\alpha_q-\tilde{X}_q\alpha_{-q})/2q\}$$

where

$$\alpha_{\pm p}=A_{\pm p}+\tilde{B}_{\pm p}C_{\pm p}\qquad \beta_{\pm p}=A_{\pm p}+\tilde{B}_{\mp p}C_{\pm p}$$

and similarly for $\alpha_{\pm q}$ and $\beta_{\pm q}$, we have formally

$$P^k_{V_\xi}=E_k(A)\qquad P^k_\eta=E_k(B)\qquad P^k_{V_\eta}=E_k(C)\qquad (k=1,2,3,4)$$

It follows that

$$\tilde{P}^k_{V_\xi}=\tilde{A}_k\qquad \tilde{P}^k_\eta=\tilde{B}_k\qquad \tilde{P}^k_{V_\eta}=\tilde{C}_k\qquad (k=1,2,3,4)$$

where $\tilde{A}_k$, $\tilde{B}_k$, and $\tilde{C}_k$ are as defined in Appendix A.

Noting

$$\tilde{r}(x)=x^3+(4\omega^2-K_2)x+2\omega K_3\qquad (x=p,-p,iq,-iq)$$

we have

$$\tilde{A}_x=(K_1x^2+2\omega K_3x+K_3^2-K_1K_2)/\tilde{r}(x)$$
$$\tilde{B}_x=(K_3x-2\omega K_1)/\tilde{r}(x)$$
$$\tilde{C}_x=(K_3x^2-2\omega K_1x)/\tilde{r}(x)$$

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